Imperial College London

Lecture 9

Poles, Zeros & Filters

(Lathi 4.10)

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Effects of Poles & Zeros on Frequency Response (1)

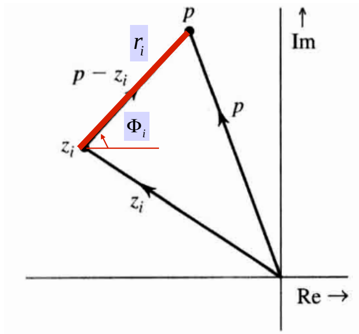
- Consider a general system transfer function:

zeros at $z_1, z_2, ..., z_N$

$$H(s) = \frac{P(s)}{Q(s)} = b_0 \frac{(s - z_1)(s - z_2)\cdots(s - z_N)}{(s - \lambda_1)(s - \lambda_2)\cdots(s - \lambda_N)}$$

Poles at λ1, λ2 …

- The value of the transfer function at some complex frequency $s = p$ is:



$$H(s)|_{s=p} = b_0 \frac{(p - z_1)(p - z_2)\cdots(p - z_N)}{(p - \lambda_1)(p - \lambda_2)\cdots(p - \lambda_N)}$$

$$= b_0 \frac{(r_1 e^{j\phi_1})(r_2 e^{j\phi_2})\cdots(r_N e^{j\phi_N})}{(d_1 e^{j\theta_1})(d_2 e^{j\theta_2})\cdots(d_N e^{j\theta_N})}$$

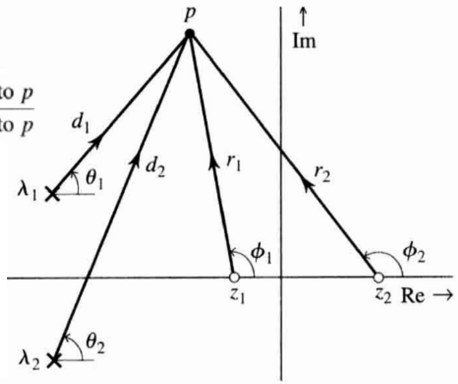
L4.10 p447

Effects of Poles & Zeros on Frequency Response (2)

- Therefore the magnitude and phase at $s = p$ are given by:

$$|H(s)|_{s=p} = b_0 \frac{r_1 r_2 \cdots r_N}{d_1 d_2 \cdots d_N}$$

$$= b_0 \frac{\text{product of the distances of zeros to } p}{\text{product of the distances of poles to } p}$$



$$\angle H(s)|_{s=p} = (\phi_1 + \phi_2 + \cdots + \phi_N) - (\theta_1 + \theta_2 + \cdots + \theta_N)$$

$$= \text{sum of zero angles to } p - \text{sum of pole angles to } p$$

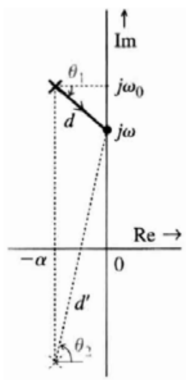
L4.10 p447

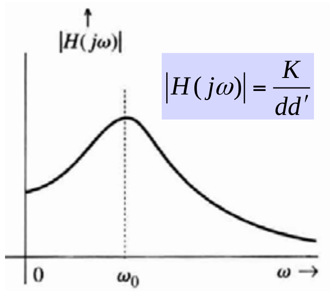
Effects of Poles & Zeros on Frequency Response (3)

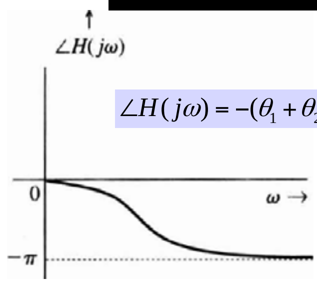
- Frequency Response of a system is obtained by **evaluating H(s) along the y-axis** (i.e. taking all value of s=jω).
- Consider the effect of two complex system poles on the frequency response.

$$H(s) = \frac{1}{(s - (\alpha + j\omega))(s - (\alpha - j\omega))}$$

Near to a pole ENHANCES amplitude







L4.10 p449

Effects of Poles & Zeros on Frequency Response (4)

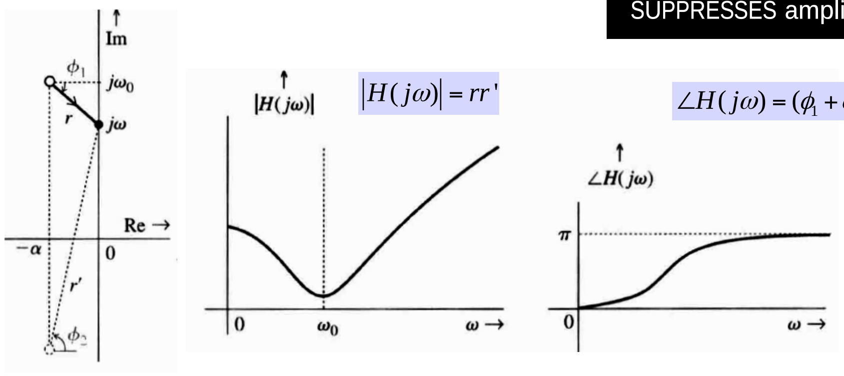
- Consider the effect of two complex system zeros on the frequency response.

$$H(s) = (s-(\alpha + j\omega))(s-(\alpha - j\omega))$$

Near to a zero SUPPRESSES amplitude

$$|H(j\omega)| = rr'$$

$$\angle H(j\omega) = (\phi_1 + \phi_2)$$



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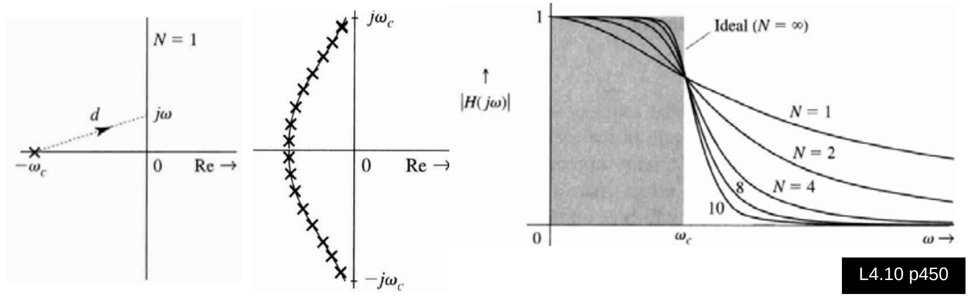
Poles & Low-pass Filters

- Use the enhancement and suppression properties of poles & zeros to design filters.
- Low-pass filter (LPF) has maximum gain at ω=0, and the gain decreases with ω.
- Simplest LPF has a single pole on real axis, say at (s=-ω_c). Then

$$H(s) = \frac{\omega_c}{s+\omega_c} \quad \text{and} \quad |H(j\omega)| = \frac{\omega_c}{d}$$

- To have a "brickwall" type of LPF (i.e. very sharp cut-off), we need a WALL OF POLE as shown, the more poles we get, the sharper the cut-off.



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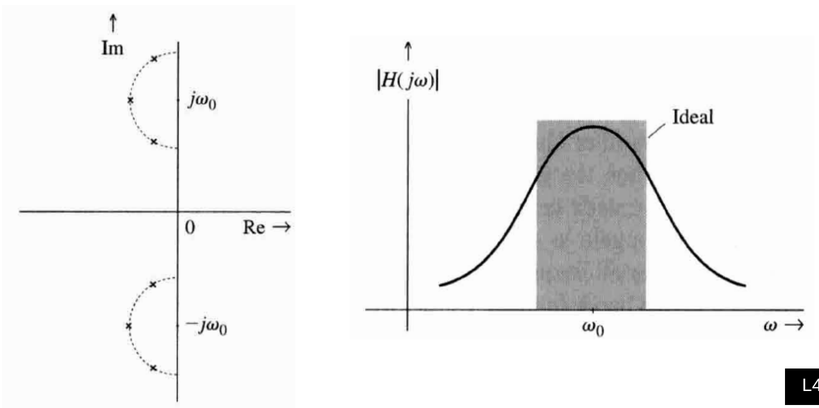
Poles & Band-pass Filter

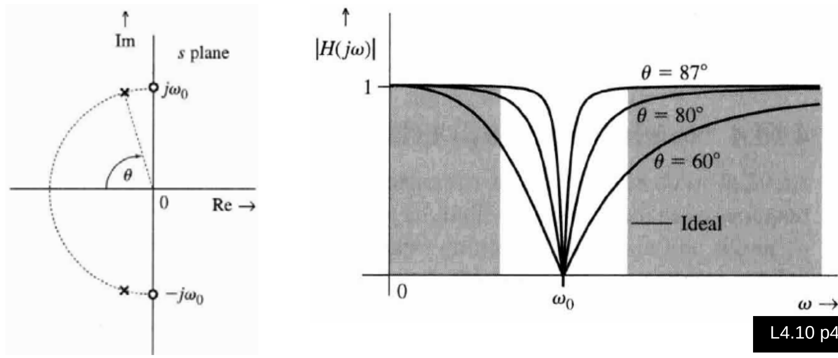
- Band-pass filter has gain enhanced over the entire passband, but suppressed elsewhere.
- For a passband centred around ω_0, we need lots of poles opposite the imaginary axis in front of the passband centre at ω_0.



L4.10 p452

Notch Filter

- Notch filter could in theory be realised with two zeros placed at $\pm j\omega_0$. However, such a filter would not have unity gain at zero frequency, and the notch will not be sharp.
- To obtain a good notch filter, put two poles close the two zeros on the semicircle as shown. Since the both pole/zero pair are equal-distance to the origin, the gain at zero frequency is exactly one. Same for $\omega=\pm\infty$.



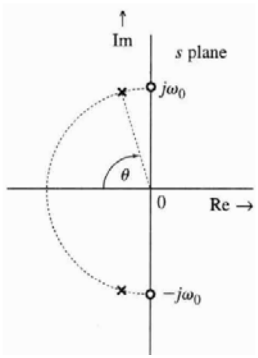
L4.10 p452

Notch Filter Example

- Design a second-order notch filter to suppress 60 Hz hum in a radio receiver.
- Make $\omega_0 = 120\pi$. Place zeros are at $s = \pm j\omega_0$, and poles at $-\omega_0 \cos\theta \pm j\omega_0 \sin\theta$.
- We get:

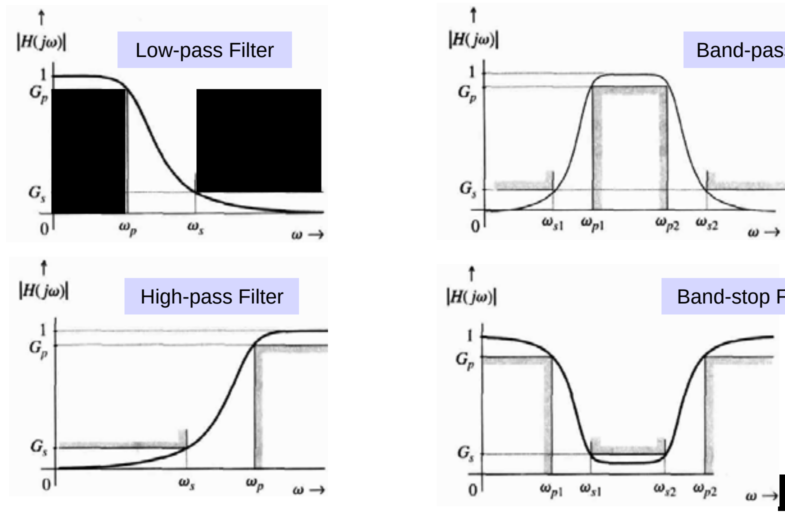
$$H(s) = \frac{(s - j\omega_0)(s + j\omega_0)}{(s + \omega_0 \cos\theta + j\omega_0 \sin\theta)(s + \omega_0 \cos\theta - j\omega_0 \sin\theta)}$$

$$= \frac{s^2 + \omega_0^2}{s^2 + (2\omega_0 \cos\theta)s + \omega_0^2} = \frac{s^2 + 142122.3}{s^2 + (753.98\cos\theta)s + 142122.3}$$



L4.10 p453

Practical Filter Specification



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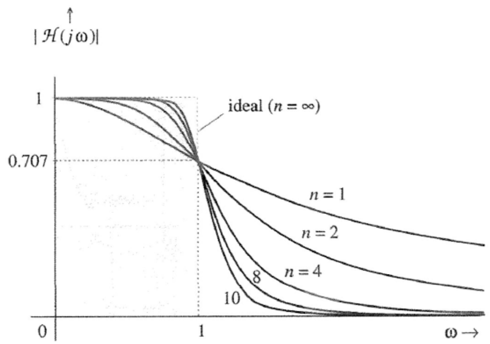
Butterworth Filters (1)

- Let us consider a normalised low-pass filter (i.e. one that has a cut-off frequency at 1) with an amplitude characteristic given by the equation:

$$|\mathcal{H}(j\omega)| = \frac{1}{\sqrt{1 + \omega^{2n}}}$$

- As $n \to \infty$, this gives a ideal LPF response: gain=1 if $\omega \le 1$, gain=0 if $\omega > 1$.



Butterworth Filters (2)

- Substitute $s = j\omega$ in the equation $|\mathcal{H}(j\omega)| = \frac{1}{\sqrt{1 + \omega^{2n}}}$
- we get: $\mathcal{H}(s)\mathcal{H}(-s) = \frac{1}{1 + (s/j)^{2n}}$
- Therefore the poles of $\mathcal{H}(s)\mathcal{H}(-s)$ are given by:

$$1 + (s/j)^{2n} = 0 \Rightarrow s^{2n} = -1 \times (j)^{2n}$$

- Now, we use the fact that $-1 = e^{j\pi(2k-1)}$ and $j = e^{j\pi/2}$ to obtain

$$s^{2n} = e^{j\pi(2k-1+n)} \qquad k \text{ integer}$$

- Therefore the poles of $\mathcal{H}(s)\mathcal{H}(-s)$ line in **unity circle** at:

$$s_k = e^{\frac{j\pi}{2n}(2k+n-1)} \qquad k = 1, 2, 3, \ldots, 2n$$

Butterworth Filters (3)

- We are only interested in H(s), not H(-s). Therefore the poles of the low-pass filter are those lying on the Left-Hand Plane (LHP) only, i.e.

$$s_k = e^{\frac{j\pi}{2n}(2k+n-1)}$$
$$= \cos\frac{\pi}{2n}(2k+n-1) + j\sin\frac{\pi}{2n}(2k+n-1) \qquad k = 1, 2, 3, \ldots, n$$

- The transfer function of this filter is:

$$\mathcal{H}(s) = \frac{1}{(s-s_1)(s-s_2)\cdots(s-s_n)}$$

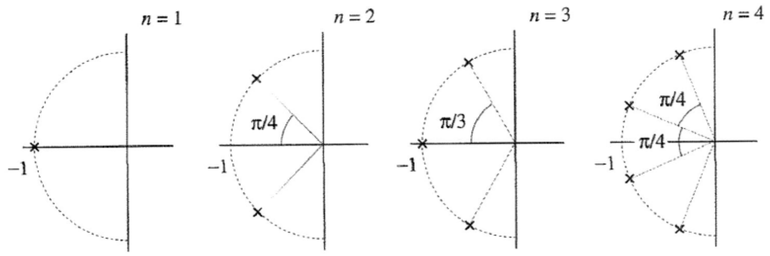
- This is a class of filter known as **Butterworth filters.**

Butterworth Filters (4)

- Butterworth filters are a family of filters with poles distributed evenly around the Left-Hand Plane (LHP) unit circle, such that the poles are given by:

$$s_k = e^{\frac{j\pi}{2n}(2k+n-1)} \quad \text{where} \quad k = 1, 2, 3, \ldots, n \quad (\text{assume } \omega_c = 1)$$

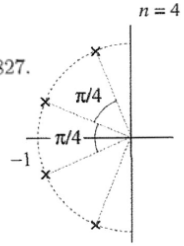
- Here are the pole locations for Butterworth filters for orders n = 1 to 4.



Butterworth Filters (5)

- Consider a fourth-order Butterworth filter (i.e. n=4).
- The poles are at angles 5π/8, 7π/8, 9π/8 amd 11π/8.
- Therefore, the pole locations are: $-0.3827 \pm j0.9239, -0.9239 \pm j0.3827.$
- Therefore:

$$\mathcal{H}(s) = \frac{1}{(s^2+0.7654s+1)(s^2+1.8478s+1)}$$
$$= \frac{1}{s^4+2.6131s^3+3.4142s^2+2.6131s+1}$$



Coefficients of Butterworth Polynomial $B_n(s) = s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + 1$

n	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9
2	1.41421356								
3	2.00000000	2.00000000							
4	2.61312593	3.41421356	2.61312593						
5	3.23606798	5.23606798	5.23606798	3.23606798					
6	3.86370331	7.46410162	9.14162017	7.46410162	3.86370331				
7	4.49395921	10.09783468	14.59179389	14.59179389	10.09783468	4.49395921			
8	5.12583090	13.13707118	21.84615097	25.68835593	21.84615097	13.13707118	5.12583090		
9	5.75877048	16.58171874	31.16343748	41.98638573	41.98638573	31.16343748	16.58171874	5.75877048	
10	6.39245322	20.43172909	42.80206107	64.88239627	74.23342926	64.88239627	42.80206107	20.43172909	6.39245322

Frequency Scaling

- So far we have consider only normalized Butterworth filters with 3dB bandwidth ω_c=1.
- We can design filters for any other cut-off frequency by substituting s by s/ ω_c.
- For example, the transfer function for a second-order Butterworth filter for ω_c=100 is given by:

$$H(s) = \frac{1}{\left(\frac{s}{100}\right)^2 + \sqrt{2}\left(\frac{s}{100}\right) + 1}$$
$$= \frac{1}{s^2 + 100\sqrt{2}s + 10^4}$$

Relating this lecture to other courses

- You will learn about poles and zeros in your 2nd year control course. The emphasis here is to provide you with intuitive understanding of their effects on frequency response.
- You have done Butterworth filters in your 2nd year analogue circuits course. Here you learn where the Butterworth filter equation comes from.
- Some of you will be implementing the notch filter in your 3rd year on real-time digital signal processor (depending on options you take), and others will learn more about filter design in your 3rd and 4th year.